Solve P.D.E.: $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$, $x>0$, $t>0$.

B.C.: $y(0,t)=0$, $t>0$

I.C.: $y(x,0)=0$, $x>0$

$\frac{\partial y}{\partial t}(x,0)=e^{-x}$, $x>0$ [106淡江機械7]

[解]將方程式左右兩邊對t取Laplace轉換

$$\int_0^\infty \frac{\partial^2 y}{\partial t^2}e^{-st}dt = \int_0^\infty 4\frac{\partial^2 y}{\partial x^2}e^{-st}dt \Rightarrow s^2Y - sy(x,0) - \frac{\partial y(x,0)}{\partial t} = 4Y_{xx}$$

$$4Y_{xx} - s^2Y = -e^{-x} \cdots\cdots\cdots(\text{i}) \Rightarrow Y_h = Ae^{sx} + Be^{-sx}$$

設$Y_p = Ce^{-x} \Rightarrow \frac{\partial Y_p}{\partial x} = -Ce^{-x} \Rightarrow \frac{\partial^2 Y_p}{\partial x^2} = Ce^{-x}$，代入(i)

$$Ce^{-x} - Cs^2e^{-x} = -e^{-x} \Rightarrow C = \frac{1}{s^2-1}$$

$$Y = Ae^{sx} + Be^{-sx} + \frac{e^{-x}}{s^2-1}\cdots\cdots\cdots(\text{ii})$$

當$x\to\infty$時，Y必須有界$\Rightarrow A=0$

邊界條件對t取Laplace轉換，得$Y(0,s)=0$，得

$$B + \frac{1}{s^2-1} = 0 \Rightarrow B = -\frac{1}{s^2-1}$$

$$Y = \frac{e^{-x}}{s^2-1} - \frac{e^{-sx}}{s^2-1}$$

而$\frac{1}{s^2-1} = \frac{1}{2}(\frac{1}{s-1} - \frac{1}{s+1})$

$$y = \frac{e^{-x}}{2}(e^t - e^{-t}) - \frac{1}{2}(e^{t-x} - e^{-t+x})u(t-x)$$

其中$u(t)$為單位階梯函數