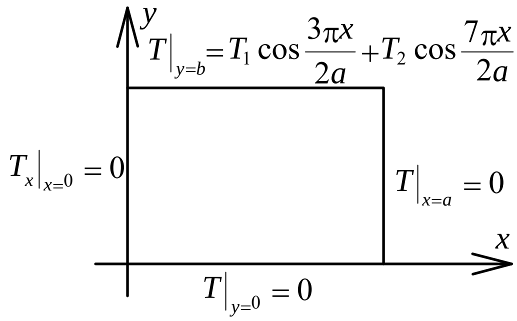
Solve the Laplace's equation $\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}=0$ with boundary conditions as shown. [105中山機電乙丙7]



[解]令$T=XY$，代入方程式$\Rightarrow X''Y+XY''=0\Rightarrow \dfrac{X''}{X}=-\dfrac{Y''}{Y}=-\lambda^2$

得兩個方程式$X''+\lambda^2 X=0,\ Y''-\lambda^2 Y=0$

由邊界條件$T(a,y)=0,\ \dfrac{\partial T(0,y)}{\partial x}=0\Rightarrow X(a)=0,\ X'(0)=0$

由X的方程式知$X=C_1\cos\lambda x+C_2\sin\lambda x$

$$\begin{cases}X(a)=0\Rightarrow C_1\cos\lambda a+C_2\sin\lambda a=0\\ X'(0)=0\Rightarrow C_2\lambda=0\end{cases}\Rightarrow C_2=0,\ \cos\lambda a=0\Rightarrow \lambda a=\frac{2n-1}{2}\pi$$

知特徵值$\lambda_n=\dfrac{2n-1}{2a}\pi\Rightarrow X_n=\cos\lambda_n x$

由Y的方程式知$Y=D_n e^{\lambda_n y}+E_n e^{-\lambda_n y}$

由邊界條件$T(x,0)=0\Rightarrow Y(0)=0\Rightarrow D_n+E_n=0\Rightarrow E_n=-D_n$

$Y=2D_n\sinh(\lambda_n y)=F_n\sinh(\lambda_n y)$

$$T(x,y)=\sum_{n=1}^{\infty}F_n\cos\lambda_n x\sinh(\lambda_n y)$$

最後的邊界條件$T(x,b)=T_1\cos\dfrac{3\pi x}{2a}+T_2\cos\dfrac{7\pi x}{2a}$代入，得

$$T_1\cos\frac{3\pi x}{2a}+T_2\cos\frac{7\pi x}{2a}=\sum_{n=1}^{\infty}F_n\cos\lambda_n x\sinh(\lambda_n b)$$

$F_2\sinh(\lambda_2 b)=T_1,\ F_4\sinh(\lambda_4 b)=T_2,$ 其他的$F_n=0$

$$T(x,y)=\frac{T_1}{\sinh(\lambda_2 b)}\cos\frac{3\pi x}{2a}\sinh\frac{3\pi y}{2a}+\frac{T_2}{\sinh(\lambda_4 b)}\cos\frac{7\pi x}{2a}\sinh\frac{7\pi y}{2a}$$